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Generalized Methodology for Assembly and Reduction of Component Models for Dynamic Substructuring

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A classic issue in component mode synthesis methods is the choice for fixed or free boundary conditions at the interface degrees of freedom and the associated vibration modes in the component's reduction basis. Closely related is the aspect of substructure assembly. In this paper a general approach to mixed boundary component mode synthesis problems is proposed, by considering component reduction and assembly separately. First, a general framework for substructure assembly is presented, enabling assembly of all types of reduced component models, regardless of their interface conditions. Thereafter, a general mixed boundary component mode synthesis method called the mixed Craig–Bampton method is introduced. The method is derived by dividing the substructure degrees of freedom into a set of internal degrees of freedom, free interface degrees of freedom, and fixed interface degrees of freedom using a simple but effective selection scheme. Based on this selection a reduction basis is computed consisting of vibration modes, constraint modes, and residual flexibility modes, which leads to relatively sparse reduced matrices. As such, the mixed Craig–Bampton method forms a natural generalization of the classic Craig–Bampton and more recent dual Craig–Bampton methods. Finally, the method is applied to a test problem, revealing that the proposed method has comparable or better accuracy and superior versatility with respect to existing methods.

Nomenclature

$\boldsymbol{A}$	=	local Boolean matrix
$\boldsymbol{g}$	=	vector of connection forces
$\boldsymbol{q}$	=	vector of generalized degrees of freedom
$\boldsymbol{B}$	=	signed Boolean matrix
$\boldsymbol{f}$	=	vector of external forces
$\boldsymbol{G}$	=	flexibility matrix
$\boldsymbol{K}$	=	stiffness matrix
$\boldsymbol{L}$	=	Boolean localization matrix
$\boldsymbol{M}$	=	mass matrix
$\boldsymbol{\eta}$	=	vector of modal coordinates
$\boldsymbol{R}$	=	reduction matrix
$\boldsymbol{u}$	=	vector of degrees of freedom
$\boldsymbol{\lambda}$	=	vector of Lagrange multipliers
$\boldsymbol{\Phi}$	=	set of vibration mode shape vectors
ϕ	=	vibration mode shape vector
ψ	=	static mode shape vector
$\boldsymbol{\Psi}$	=	set of static mode shape vectors
ω	=	eigenfrequency
$\tilde{\star}$	=	stiffness reduced matrix/vector
$\bar{\star}$	=	flexibility reduced matrix/vector

Superscript

(s)	=	pertaining to a structure s

I. Introduction

ALTHOUGH the first dynamic substructuring (DS) techniques have been developed over four decades ago, it is still an active field of research. DS techniques use the old principle of "divide and conquer" to solve structural dynamic problems by first dividing the system at hand in components, then determining the dynamics of the components and subsequently assembling these to find the dynamics of the complete system. Despite tremendous advances in computation power and algorithm efficiency, analyzing a system's structural dynamics in a componentwise fashion has proven to have some important advantages over global methods where the entire problem is handled at once, namely:

1) Evaluation of the dynamic behavior of structures too large or complex to be analyzed as a whole.

2) Local dynamic behavior can be recognized more easily than when the entire system is analyzed.

3) It enables to combine modeled parts and experimentally identified components.

4) It allows sharing and combining substructures from different project groups.

In this paper we adopt the vision outlined in [1] and refer to "dynamic substructuring" as any technique that tries to find the structural dynamics (either in the time, "modal" (reduced) or frequency domain) of a system through assembly of the dynamics of its components. Historically, the first dynamic substructuring ideas were developed as model reduction techniques and were probably triggered by the paper of Hurty in 1960 [2] and further worked out in [3]. These methods were soon known under the name "component mode synthesis" (CMS), where the term "modes" can refer to all kinds of structural shape vectors. In the view described above, these component mode synthesis techniques in fact constitute a specific class of substructuring techniques where component models are first reduced, using some CMS reduction basis, and subsequently assembled. Following the work of Hurty [2], the scientific and engineering communities rapidly discovered the benefits of dynamic substructuring and component mode synthesis became an important research topic in the field of structural dynamics. Some major developments followed shortly, resulting in the classic methods by Craig and Bampton [4], MacNeal [5], and Rubin [6] in the late 1960s and early 1970s. For a more detailed overview of dynamic substructuring techniques in general see for instance [1]; for an overview of CMS methods in specific we refer to chapter 17 of [7].

CMS methods can be classified according to many criteria, but one classification often seen in the literature (e.g., in [8]) is the distinction made between "fixed interface" and "free interface" methods. To be more precise, this classification refers to the type of vibration modes used in the CMS reduction basis, namely either employing vibration modes of the component fixed at its interface or vibration modes with the component interface left free. The most well-known variant of the

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former class is the Craig–Bampton method, while the MacNeal and Rubin methods ([5,6]), as well as the more recent dual Craig–Bampton (DCB) method [9], are examples of the latter class.

From the above classification a question that naturally arises is: when should one use fixed interface methods and when are free interface methods to be preferred? This question is relevant when we imagine a situation where two components are assembled: component one is stiff and/or heavy whereas substructure two is flexible and/or light. One can imagine that after assembly the stiff structure will behave as if its interface is still quasi free, thereby largely dictating the motion of the flexible structure through its interface. Ideally, one would thus reduce substructure one using a free interface mode and component two using fixed interface modes. However, this can still lead to difficulties where one of the substructures is for example very stiff in one direction/at one location but relatively flexible in another direction/location. In this case one would want to employ a mix of both methods per substructure, that is, fix some interface degrees of freedom in the model reduction while others are left free.

In the past, several mixed boundary CMS methods have been proposed [10–12]. In those contributions, component model reduction and component assembly are treated in an integral way; reduced component models are transformed such that they can be assembled using the standard CMS ("superelement") assembly technique. Here, however, we take a different approach by considering separately component assembly and model reduction. In this way both component assembly and reduction are performed in a truly mixed sense, an idea first outlined in [13] in an ad hoc fashion.

The remainder of the paper is therefore organized as follows. First, we will introduce a general framework for substructure assembly techniques as this is a prerequisite to assemble mixed boundary component models. Thereafter, Sec. III reviews the classic Craig–Bampton (CB) and recent DCB component model reduction methods. This paves the way for the derivation of the mixed Craig–Bampton (MCB) mixed boundary component reduction method in Sec. IV. Section V then covers a case study of the different methods on a test problem; Sec. VI ends the paper with some conclusions.

II. General Framework for Substructure Assembly

This section addresses an important aspect of dynamic substructuring, namely the assembly of component models to obtain the structural dynamic model of the total system. In the majority of the literature on DS, however, component assembly is treated as an integral aspect of the method and one simply assembles the (reduced) component as one would assemble any other finite element. In this section, we will present a general framework for substructure assembly, needed to assemble the MCB reduced systems that will be introduced in the next section.

In general, structural dynamic component models can be expressed either in terms of stiffness or flexibility, or a mix of both, at their interface degrees of freedom. The former is the case for regular full finite element (FE) models as well as models reduced using classical CMS methods (such as the Craig–Bampton method), while the latter is the case for *dual* or *mixed* reduced systems resulting from the dual and mixed Craig–Bampton methods, (see Secs. III.B and IV), respectively. Hence, the substructure degree-of-freedom (DOF) vector either contains some set of internal DOF and *interface displacements* or some set of internal DOF and *interface forces*. Given the different representations of the substructure models on the interface, three assembly cases can be distinguished:

1) Assembly of interface displacements to interface displacements, or "stiffness assembly."

2) Assembly of interface forces to interface forces, or "flexibility assembly."

3) Assembly of interface displacements to interface forces, or "mixed assembly."

These three cases are illustrated in Fig. 1 and will be treated in detail in the subsequent sections, where we will use a so-called three-field variational formulation to derive the required assembly procedures. Such an approach is needed to tackle the mixed assembly problem; for the more straightforward cases of stiffness and flexibility assembly a two-field formulation is already sufficient.

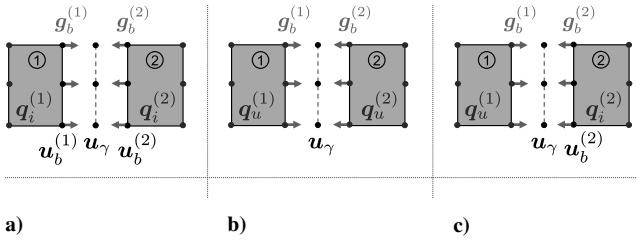

Fig. 1 Three different assembly cases.

Before we proceed with the actual treatment of the assembly procedures, let us first introduce some notations. Vectors $\boldsymbol{u}$ denote physical displacements, while a vector $\boldsymbol{q}$ denotes a set of generalized DOF. A superscript $\tilde{\star}$ denotes a reduced substructure with displacement interface DOF (i.e., stiffness interface representation) and $\bar{\star}$ denotes a reduced substructure with force interface DOF (i.e., flexibility interface representation).

A. Stiffness Assembly

In this section we treat the assembly of components expressed in terms of stiffness at the structural interface. As indicated before, this can either be a full FE model, or an FE model reduced using any of the classic CMS methods. We start from the linear, discretized and possibly reduced equations of motion of a substructure s connected to other substructures:

$$\boldsymbol{M}^{(s)}\ddot{\boldsymbol{q}}^{(s)} + \boldsymbol{K}^{(s)}\boldsymbol{q}^{(s)} = \boldsymbol{f}^{(s)} + \boldsymbol{g}^{(s)}$$

Here $\boldsymbol{M}^{(s)}$ denotes the substructure's mass matrix, $\boldsymbol{K}^{(s)}$ is the stiffness matrix, $\boldsymbol{q}^{(s)}$ some vector of generalized degrees of freedom, $\boldsymbol{f}^{(s)}$ the external excitation vector and $\boldsymbol{g}^{(s)}$ the vector of connection forces felt from connected substructures. We partition the subsystem DOF (and the associated matrices) in an internal subscript i and boundary subscript b part and write the generalized DOF vector and vector of connection forces as:

$$\boldsymbol{q}^{(s)} = \begin{bmatrix} \boldsymbol{q}_i^{(s)} \\ \boldsymbol{u}_b^{(s)} \end{bmatrix}, \qquad \boldsymbol{g}^{(s)} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{g}_b^{(s)} \end{bmatrix}$$

Here $\boldsymbol{q}_i$ are the generalized internal DOF (for nonreduced systems these are simply the nodal displacements $\boldsymbol{u}_i$) and $\boldsymbol{u}_b$ the physical boundary displacement DOF. By definition the vector of connection forces $\boldsymbol{g}$ is zero at the internal subsystem DOF, while $\boldsymbol{g}_b^{(s)}$ is similar to the local Lagrange multipliers used for instance in the algebraic finite elements tearing and interconnecting (FETI) method [14]. Between the substructures we can define the intermediate interface displacement field $\boldsymbol{u}_\gamma$, to govern the compatibility of substructural displacements at the interface. This condition then writes:

$$\boldsymbol{u}_b^{(s)} - \boldsymbol{L}_b^{(s)}\boldsymbol{u}_\gamma = \boldsymbol{0} \tag{1}$$

Here $\boldsymbol{L}_b^{(s)}$ is a Boolean matrix localizing the DOF from the global intermediate displacement field corresponding to the substructure boundary DOF. As a result, we end up with a three-field formulation of the substructuring problem, having as independent unknowns the substructure DOF field $\boldsymbol{q}^{(s)}$, the field of interface connection forces $\boldsymbol{g}_b^{(s)}$ and the intermediate interface displacement field $\boldsymbol{u}_\gamma$. Taking a variational approach we can now obtain the assembled equations. To this end, we can set up the Lagrangian of this problem as:

$$\mathcal{L}(\boldsymbol{q}^{(s)}, \boldsymbol{g}_b^{(s)}, \boldsymbol{u}_\gamma) = \sum_s \left(\frac{1}{2}\boldsymbol{q}^{(s)T}\boldsymbol{K}^{(s)}\boldsymbol{q}^{(s)} - \boldsymbol{f}^{(s)T}\boldsymbol{q}^{(s)} + \boldsymbol{g}_b^{(s)T}\left(\boldsymbol{L}_b^{(s)}\boldsymbol{u}_\gamma - \boldsymbol{u}_b^{(s)}\right)\right) \tag{2}$$

Note that the above considers the static problem only. To include the inertia terms, one needs to go back to the variational principle in

terms of the Hamiltonian [15]; these details will not be treated here. However, since in Eq. (1) we chose to express the compatibility condition in terms of displacements (instead of accelerations) the coupling between substructure will occur in the assembled stiffness matrix. The assembled mass matrix is hence not of primary interest.

For the sake of simplicity we will now consider the assembly problem for two substructures only ($s = 1, 2$), although the concepts presented in the coming sections can be easily generalized to the assembly of an arbitrary number of components. This situation is depicted in Fig. 1a. To find the assembled equations we take the variation with respect to the free variables to find the equations of motion of the assembly of two components:

$$\begin{bmatrix} \boldsymbol{K}_{ii}^{(1)} & \boldsymbol{K}_{ib}^{(1)} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{K}_{bi}^{(1)} & \boldsymbol{K}_{bb}^{(1)} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{K}_{ii}^{(2)} & \boldsymbol{K}_{ib}^{(2)} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{K}_{bi}^{(2)} & \boldsymbol{K}_{bb}^{(2)} & \boldsymbol{0} & -\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{L}_b^{(1)} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{L}_b^{(2)} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{L}_b^{(1)^T} & \boldsymbol{L}_b^{(2)^T} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_i^{(1)} \\ \boldsymbol{u}_b^{(1)} \\ \boldsymbol{q}_i^{(2)} \\ \boldsymbol{u}_b^{(2)} \\ \boldsymbol{g}_b^{(1)} \\ \boldsymbol{g}_b^{(2)} \\ \boldsymbol{u}_\gamma \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_i^{(1)} \\ \boldsymbol{f}_b^{(1)} \\ \boldsymbol{f}_i^{(2)} \\ \boldsymbol{f}_b^{(2)} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} \tag{3}$$

In the above equations we can recognize the fifth and sixth equations as the compatibility conditions, governing the compatibility between the $\boldsymbol{u}_b^{(s)}$ and $\boldsymbol{u}_\gamma$. The last row is the equilibrium condition on the interface, stating that the sum of the substructure connection forces must be zero.

For completeness, the assembled mass matrix is shown below; it is simply a block diagonal matrix containing the component matrices. With the mass matrix, the complete dynamic equations of the assembled system are found. However, these assembled equations of motion still contain the full three fields, which is for most analyses inefficient from a computational point of view.[†] Therefore, it is desired to simplify the equations. In essence two ways exist to do this, namely the so-called *primal* or *dual* assembly, as discussed next

$$\boldsymbol{M}^{(\mathrm{tot})} = \begin{bmatrix} \boldsymbol{M}_{ii}^{(1)} & \boldsymbol{M}_{ib}^{(1)} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{M}_{bi}^{(1)} & \boldsymbol{M}_{bb}^{(1)} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{M}_{ii}^{(2)} & \boldsymbol{M}_{ib}^{(2)} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{M}_{bi}^{(2)} & \boldsymbol{M}_{bb}^{(2)} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}$$

1. *Dual Assembly*

In dual assembly, one eliminates the interface connection force fields by realizing that the interface forces should be equal and opposite to satisfy the interface equilibrium. To a priori satisfy this condition, we can introduce a unique field of interface forces $\boldsymbol{\lambda}$, as follows:

$$\boldsymbol{g}_b^{(s)} = -\boldsymbol{B}_b^{(s)^T} \boldsymbol{\lambda}$$

Here $\boldsymbol{B}_b^{(s)}$ is a signed Boolean matrix acting on the substructure interface DOF and $\boldsymbol{\lambda}$ corresponds physically to the interface force intensities. Note that the minus sign is chosen to stress the fact that whereas $\boldsymbol{g}_b^{(s)}$ was seen as an external force for the substructure, $\boldsymbol{\lambda}$ is considered an internal force. Because of the construction of the Boolean matrices it holds that [1]

$$\sum_s \boldsymbol{B}_b^{(s)} \boldsymbol{L}_b^{(s)} = \boldsymbol{0} \tag{4}$$

Hence, this choice for the interface connection forces satisfies the interface equilibrium for any $\boldsymbol{\lambda}$. This choice gives rise to the following transformation:

$$\begin{bmatrix} \boldsymbol{q}_i^{(1)} \\ \boldsymbol{u}_b^{(1)} \\ \boldsymbol{q}_i^{(2)} \\ \boldsymbol{u}_b^{(2)} \\ \boldsymbol{g}_b^{(1)} \\ \boldsymbol{g}_b^{(2)} \\ \boldsymbol{u}_\gamma \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{B}_b^{(1)^T} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{B}_b^{(2)^T} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_i^{(1)} \\ \boldsymbol{u}_b^{(1)} \\ \boldsymbol{q}_i^{(2)} \\ \boldsymbol{u}_b^{(2)} \\ \boldsymbol{\lambda} \\ \boldsymbol{u}_\gamma \end{bmatrix} \tag{5}$$

Substituting this transformation in the three-field assembled equations of motion (3) replaces the local connection forces by the unique global field $\boldsymbol{\lambda}$, satisfying the interface equilibrium condition.[‡] To end up with a symmetric system one can use the above transformation to subsequently premultiply the equations. This eliminates the intermediate displacement field $\boldsymbol{u}_\gamma$, due to the relation between the Boolean matrices in Eq. (4). The procedure is illustrated in Fig. 2 and the simplified assembled equations become:

$$\cdots + \begin{bmatrix} \boldsymbol{K}_{ii}^{(1)} & \boldsymbol{K}_{ib}^{(1)} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{K}_{bi}^{(1)} & \boldsymbol{K}_{bb}^{(1)} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{B}_b^{(1)^T} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{K}_{ii}^{(2)} & \boldsymbol{K}_{ib}^{(2)} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{K}_{bi}^{(2)} & \boldsymbol{K}_{bb}^{(2)} & \boldsymbol{B}_b^{(2)^T} \\ \boldsymbol{0} & \boldsymbol{B}_b^{(1)} & \boldsymbol{0} & \boldsymbol{B}_b^{(2)} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_i^{(1)} \\ \boldsymbol{u}_b^{(1)} \\ \boldsymbol{q}_i^{(2)} \\ \boldsymbol{u}_b^{(2)} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_i^{(1)} \\ \boldsymbol{f}_b^{(1)} \\ \boldsymbol{f}_i^{(2)} \\ \boldsymbol{f}_b^{(2)} \\ \boldsymbol{0} \end{bmatrix} \tag{6}$$

The mass matrix has been omitted for compactness and is simply the block diagonal matrix of component mass matrices. The above system is called the *dual* assembled system, since the unknowns defining the interface problem are forces which are mathematically dual to the original displacement unknowns. As a result, the compatibility condition is present explicitly in the assembled equations of motion, i.e., the last row in Eq. (6). Note that dual assembly approaches were already considered in the early days of finite element theory, but only became popular in the 1990s as a way to implement efficient solvers on parallel processing computers. This led to the family of parallel solvers known as FETI [17].

2. *Primal Assembly*

An even further simplified expression for the assembled system can be obtained by realizing that the compatibility condition can be a priori satisfied by choosing a unique set of substructure interface DOF as:

$$\boldsymbol{u}_b^{(s)} = \boldsymbol{L}_b^{(s)} \boldsymbol{u}_\gamma$$

This choice gives the following transformation:

$$\begin{bmatrix} \boldsymbol{q}_i^{(1)} \\ \boldsymbol{u}_b^{(1)} \\ \boldsymbol{q}_i^{(2)} \\ \boldsymbol{u}_b^{(2)} \\ \boldsymbol{g}_b^{(1)} \\ \boldsymbol{g}_b^{(2)} \\ \boldsymbol{u}_\gamma \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{L}_b^{(1)} \\ \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{L}_b^{(2)} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_i^{(1)} \\ \boldsymbol{q}_i^{(2)} \\ \boldsymbol{g}_b^{(1)} \\ \boldsymbol{g}_b^{(2)} \\ \boldsymbol{u}_\gamma \end{bmatrix}$$

Again, this transformation is substituted in Eq. (3) thereby eliminating the substructure boundary DOF sets $\boldsymbol{u}_b^{(s)}$ and satisfying the interface compatibility condition. Premultiplication is then needed to obtain a symmetric system of equations. Using again the relation between the Boolean matrices $\boldsymbol{L}_b$ and $\boldsymbol{B}_b$ in Eq. (4), we see that by doing so the equilibrium condition is also satisfied and drops

[†]The three-field formulation might have some advantages, for instance in formulating contact problems between nonmatching grids [16].

[‡]Note that although the name "dual Craig–Bampton" was coined in [9], a similar but less general method was also discussed in [13].

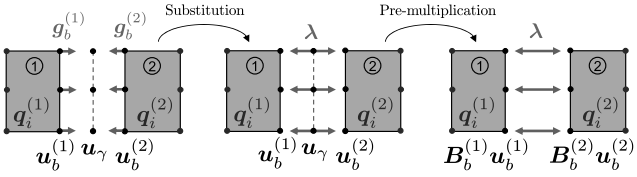

Fig. 2 Simplification of the three-field formulation for stiffness assembly according to dual assembly.

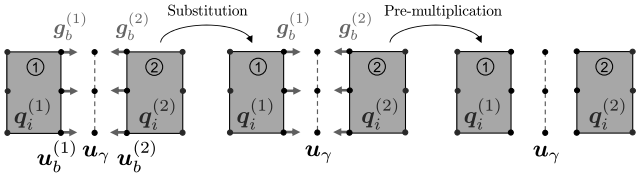

Fig. 3 Simplification of the three-field formulation for stiffness assembly according to primal assembly.

out of the equation. The procedure is illustrated in Fig. 3 and results in the following expression for the assembled system:

$$\cdots + \begin{bmatrix} \boldsymbol{K}_{ii}^{(1)} & \boldsymbol{0} & \boldsymbol{K}_{ib}^{(1)}\boldsymbol{L}_b^{(1)} \\ \boldsymbol{0} & \boldsymbol{K}_{ii}^{(2)} & \boldsymbol{K}_{ib}^{(2)}\boldsymbol{L}_b^{(2)} \\ \boldsymbol{L}_b^{(1)^T}\boldsymbol{K}_{bi}^{(1)} & \boldsymbol{L}_b^{(2)^T}\boldsymbol{K}_{bi}^{(2)} & \boldsymbol{L}_b^{(1)^T}\boldsymbol{K}_{bb}^{(1)}\boldsymbol{L}_b^{(1)} + \boldsymbol{L}_b^{(2)^T}\boldsymbol{K}_{bb}^{(2)}\boldsymbol{L}_b^{(2)} \end{bmatrix} \times \begin{bmatrix} \boldsymbol{q}_i^{(1)} \\ \boldsymbol{q}_i^{(2)} \\ \boldsymbol{u}_\gamma \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_i^{(1)} \\ \boldsymbol{f}_i^{(2)} \\ \boldsymbol{L}_b^{(1)^T}\boldsymbol{f}_b^{(1)} + \boldsymbol{L}_b^{(2)^T}\boldsymbol{f}_b^{(2)} \end{bmatrix} \tag{7}$$

The mass matrix is again omitted for compactness but is similar to the stiffness matrix. The above equations form the so-called *primal* assembled system; the most compact form of the assembled equations of motion using a minimum number of DOF. Note that this type of assembly is the way individual elements are classically assembled in a finite element method. Furthermore primal assembly is almost always used in component mode synthesis methods.

3. Mix of Both: Dirichlet–Neumann Assembly

In addition to the primal and dual assembly methods treated above, the assembled equations in Eq. (3) can also be simplified by combining both methods, that is, by choosing both a unique set of interface DOF and interface forces. Thereby, both the equilibrium and compatibility condition on the interface are satisfied a priori. This gives rise to the following transformation:

$$\begin{bmatrix} \boldsymbol{q}_i^{(1)} \\ \boldsymbol{u}_b^{(1)} \\ \boldsymbol{q}_i^{(2)} \\ \boldsymbol{u}_b^{(2)} \\ \boldsymbol{g}_b^{(1)} \\ \boldsymbol{g}_b^{(2)} \\ \boldsymbol{u}_\gamma \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{L}_b^{(1)} \\ \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{L}_b^{(2)} \\ \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{B}_b^{(1)^T} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{B}_b^{(2)^T} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_i^{(1)} \\ \boldsymbol{q}_i^{(2)} \\ \boldsymbol{\lambda} \\ \boldsymbol{u}_\gamma \end{bmatrix}$$

Substitution of this transformation in the three-field assembled equations of motion in Eq. (3) simultaneously introduces the interface force field $\boldsymbol{\lambda}$ and eliminates the substructure boundary DOF $\boldsymbol{u}_b^{(s)}$, as illustrated in Fig. 4. This gives:

$$\begin{bmatrix} \boldsymbol{M}_{ii}^{(1)} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{M}_{ib}^{(1)}\boldsymbol{L}_b^{(1)} \\ \boldsymbol{M}_{bi}^{(1)} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{M}_{bb}^{(1)}\boldsymbol{L}_b^{(1)} \\ \boldsymbol{0} & \boldsymbol{M}_{ii}^{(2)} & \boldsymbol{0} & \boldsymbol{M}_{ib}^{(2)}\boldsymbol{L}_b^{(2)} \\ \boldsymbol{0} & \boldsymbol{M}_{bi}^{(2)} & \boldsymbol{0} & \boldsymbol{K}_{bb}^{(2)}\boldsymbol{L}_b^{(2)} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{q}}_i^{(1)} \\ \ddot{\boldsymbol{q}}_i^{(2)} \\ \ddot{\boldsymbol{\lambda}} \\ \ddot{\boldsymbol{u}}_\gamma \end{bmatrix} + \cdots \begin{bmatrix} \boldsymbol{K}_{ii}^{(1)} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{K}_{ib}^{(1)}\boldsymbol{L}_b^{(1)} \\ \boldsymbol{K}_{bi}^{(1)} & \boldsymbol{0} & -\boldsymbol{B}_b^{(1)^T} & \boldsymbol{K}_{bb}^{(1)}\boldsymbol{L}_b^{(1)} \\ \boldsymbol{0} & \boldsymbol{K}_{ii}^{(2)} & \boldsymbol{0} & \boldsymbol{K}_{ib}^{(2)}\boldsymbol{L}_b^{(2)} \\ \boldsymbol{0} & \boldsymbol{K}_{bi}^{(2)} & -\boldsymbol{B}_b^{(2)^T} & \boldsymbol{K}_{bb}^{(2)}\boldsymbol{L}_b^{(2)} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_i^{(1)} \\ \boldsymbol{q}_i^{(2)} \\ \boldsymbol{\lambda} \\ \boldsymbol{u}_\gamma \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_i^{(1)} \\ \boldsymbol{f}_b^{(1)} \\ \boldsymbol{f}_i^{(2)} \\ \boldsymbol{f}_b^{(2)} \end{bmatrix} \tag{8}$$

where three lines of zeros have dropped out of the equation, resulting in a square system of equations. As can be seen this expression for the assembled system is nonsymmetric, therefore, in practice this form of the assembled equations is often not very useful. Premultiplication with this transformation matrix does solve this, as it eliminates the Lagrange multipliers and results in the primal assembled system obtained earlier. The reason we still show this type of assembly is that a similar transformation is used in the case of mixed assembly. Note that the name of this type of assembly refers to the way the assembled system can be solved, namely using the Gauss–Seidel method. This leads to so-called Dirichlet–Neumann iterations known from domain decomposition theory [18].

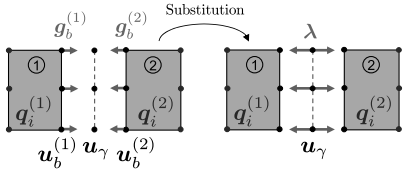

Fig. 4 Simplification of the three-field formulation for stiffness assembly using to Dirichlet–Neumann assembly.

B. Flexibility Assembly

In this section we will address the assembly of two subsystems reduced such that they are expressed in terms of flexibility at their interface, resulting for example from a DCB reduction. For such systems, the original displacement field $\boldsymbol{u}^{(s)}$ of the substructure is approximated by some set of generalized DOF and interface forces, as:

$$\boldsymbol{u}^{(s)} = \bar{\boldsymbol{R}}^{(s)}\bar{\boldsymbol{q}}^{(s)} = \bar{\boldsymbol{R}}^{(s)}\begin{bmatrix} \boldsymbol{q}_u^{(s)} \\ \boldsymbol{g}_b^{(s)} \end{bmatrix} \tag{9}$$

To come to the assembled equations of motion for multiple components we need to formulate again the Lagrangian of the substructuring problem. However due to the reduction, the boundary DOF are no longer explicitly present in the reduced substructure descriptions. Hence we write

$$\boldsymbol{u}_b^{(s)} = \bar{\boldsymbol{R}}_b^{(s)}\bar{\boldsymbol{q}}^{(s)}$$

where $\bar{\boldsymbol{R}}_b^{(s)}$ is the part of the reduction matrix associated with the boundary DOF. This then allows to set up the Lagrangian of this problem as:

$$\mathcal{L}(\bar{\boldsymbol{q}}^{(s)}, \bar{\boldsymbol{g}}_b^{(s)}, \boldsymbol{u}_\gamma) = \sum_s \left(\frac{1}{2}\bar{\boldsymbol{q}}^{(s)^T}\bar{\boldsymbol{K}}^{(s)}\bar{\boldsymbol{q}}^{(s)} - \bar{\boldsymbol{f}}^{(s)^T}\bar{\boldsymbol{q}}^{(s)} + \boldsymbol{g}_b^{(s)^T}\left(\boldsymbol{L}_b^{(s)}\boldsymbol{u}_\gamma - \bar{\boldsymbol{R}}_b^{(s)}\bar{\boldsymbol{q}}^{(s)}\right)\right) \tag{10}$$

where we have

$$\bar{\boldsymbol{K}}^{(s)} = \bar{\boldsymbol{R}}^{(s)^T}\boldsymbol{K}^{(s)}\bar{\boldsymbol{R}}^{(s)} \quad \text{and} \quad \bar{\boldsymbol{f}}^{(s)} = \bar{\boldsymbol{R}}^{(s)^T}\boldsymbol{f}^{(s)}$$

As before we now consider the assembly of two subsystems for the sake of illustration, as shown schematically in Fig. 1b. Taking again

the variation of this expression to the free variables, one obtains the assembled equations of motion as:

$$\cdots + \begin{bmatrix} \bar{\boldsymbol{K}}_{uu}^{(1)} & \bar{\boldsymbol{K}}_{ug}^{(1)} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \bar{\boldsymbol{K}}_{gu}^{(1)} & \bar{\boldsymbol{K}}_{gg}^{(1)} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{L}_b^{(1)} \\ \boldsymbol{0} & \boldsymbol{0} & \bar{\boldsymbol{K}}_{uu}^{(2)} & \bar{\boldsymbol{K}}_{ug}^{(2)} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \bar{\boldsymbol{K}}_{gu}^{(2)} & \bar{\boldsymbol{K}}_{gg}^{(2)} & \boldsymbol{L}_b^{(2)} \\ \boldsymbol{0} & \boldsymbol{L}_b^{(1)^T} & \boldsymbol{0} & \boldsymbol{L}_b^{(2)^T} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_u^{(1)} \\ \boldsymbol{g}_b^{(1)} \\ \boldsymbol{q}_u^{(2)} \\ \boldsymbol{g}_b^{(2)} \\ \boldsymbol{u}_\gamma \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_u^{(1)} \\ \bar{\boldsymbol{f}}_b^{(1)} \\ \boldsymbol{f}_u^{(2)} \\ \bar{\boldsymbol{f}}_b^{(2)} \\ \boldsymbol{0} \end{bmatrix} \tag{11}$$

In the above equation, the contribution of $\bar{\boldsymbol{K}}_b^{(s)}$ is implicitly present in the reduced matrices $\bar{\boldsymbol{K}}_{ug}^{(s)}$, $\bar{\boldsymbol{K}}_{gu}^{(s)}$ and $\bar{\boldsymbol{K}}_{gg}^{(s)}$. The mass matrix is not shown for compactness; it is again a block diagonal matrix. Note that the part $\bar{\boldsymbol{K}}_{gg}^{(s)}$ of the stiffness matrix acting on the interface DOF is in fact a true flexibility matrix and the external excitations $\bar{\boldsymbol{f}}_b^{(s)}$ are in fact interface displacements due to the applied external forces $\boldsymbol{f}_b^{(s)}$. See Secs. III and III.B for the full expressions of those submatrices for the case of the DCB method.

Note that although we started from a three-field formulation, the above equation is actually the dual assembled form for interface flexibility type of substructures, as can be seen by comparing Eqs. (11) and (6). In contrast to the stiffness assembly of the previous section, here the dual DOF are the interface connection forces, while the intermediate interface displacements constitute the unique field. Because of the reduction, the compatibility conditions for the components are stated in the second and fourth rows in a weakened form; see Sec. III.B. The fifth row can be recognized as the equilibrium condition.

From the above set of assembled equations, the only simplification to be made is therefore to go to a true primal system as in the previous section. As in the dual assembly of the previous section, we again choose the interface forces in the form:

$$\boldsymbol{g}_b^{(s)} = -\boldsymbol{B}_b^{(s)^T} \boldsymbol{\lambda}$$

As before, this gives the following transformation:

$$\begin{bmatrix} \boldsymbol{q}_u^{(1)} \\ \boldsymbol{g}_b^{(1)} \\ \boldsymbol{q}_u^{(2)} \\ \boldsymbol{g}_b^{(2)} \\ \boldsymbol{u}_\gamma \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{B}_b^{(1)^T} \\ \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{B}_b^{(2)^T} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_u^{(1)} \\ \boldsymbol{q}_u^{(2)} \\ \boldsymbol{u}_\gamma \\ \boldsymbol{\lambda} \end{bmatrix}$$

Again, we first substitute the above transformation in Eq. (11). This introduces the unique interface force field $\boldsymbol{\lambda}$ and eliminates the $\boldsymbol{g}_b^{(s)}$, hence the equilibrium condition is satisfied. Premultiplication then eliminates the interface displacement field (illustrated in Fig. 5) and gives the primal assembled system as:

$$\cdots + \begin{bmatrix} \bar{\boldsymbol{K}}_{uu}^{(1)} & \boldsymbol{0} & \bar{\boldsymbol{K}}_{ug}^{(1)} \boldsymbol{B}_b^{(1)^T} \\ \boldsymbol{0} & \bar{\boldsymbol{K}}_{uu}^{(2)} & \bar{\boldsymbol{K}}_{ug}^{(2)} \boldsymbol{B}_b^{(2)^T} \\ \boldsymbol{B}_b^{(1)} \boldsymbol{K}_{gu}^{(1)} & \boldsymbol{B}_b^{(2)} \bar{\boldsymbol{K}}_{gu}^{(2)} & \boldsymbol{B}_b^{(1)} \bar{\boldsymbol{K}}_{gg}^{(1)} \boldsymbol{B}_b^{(1)^T} + \boldsymbol{B}_b^{(2)} \bar{\boldsymbol{K}}_{gg}^{(2)} \boldsymbol{B}_b^{(2)^T} \end{bmatrix} \times \begin{bmatrix} \boldsymbol{q}_u^{(1)} \\ \boldsymbol{q}_u^{(2)} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_u^{(1)} \\ \boldsymbol{f}_u^{(2)} \\ \boldsymbol{B}_b^{(1)} \bar{\boldsymbol{f}}_b^{(1)} + \boldsymbol{B}_b^{(2)} \bar{\boldsymbol{f}}_b^{(2)} \end{bmatrix} \tag{12}$$

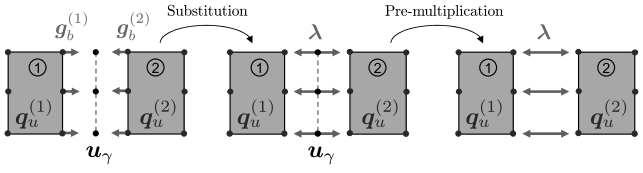

Fig. 5 Simplification of the three-field formulation of the flexibility assembly case.

Here the assembled mass matrix is again not shown; it is similar to the stiffness matrix. As we can see, this primal form of the equations of motion is very similar to that of the previous section, consisting only of the generalized substructure DOF and one unique interface field. Because of the flexibility expression of the substructure interfaces the unique interface field here is the set of interface forces, whereas in the previous section it was the unique set of interface displacements.

C. Mixed Assembly

In this section we consider the case of mixed assembly, where one subsystem is expressed in terms of stiffness at its interface while the other interface is expressed in terms of flexibility. This is the case when assembling a mix of Craig–Bampton and DCB reduced systems, or when assembling MCB reduced systems. To derive the assembled equations of motion in this case, one simply combines the Lagrangians found earlier for the stiffness and flexibility assembly cases in Eqs. (2) and (10) and takes their variation.

Let us consider the case where we want to assemble substructure 1, expressed in flexibility at its interface, with component 2, which has an interface stiffness representation. This is schematically shown in Fig. 1c. The following three-field assembled equations are then found from a variational approach:

$$\cdots + \begin{bmatrix} \bar{\boldsymbol{K}}_{uu}^{(1)} & \bar{\boldsymbol{K}}_{ug}^{(1)} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \bar{\boldsymbol{K}}_{gu}^{(1)} & \bar{\boldsymbol{K}}_{gg}^{(1)} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{L}_b^{(1)} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{K}_{ii}^{(2)} & \boldsymbol{K}_{ib}^{(2)} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{K}_{bi}^{(2)} & \boldsymbol{K}_{bb}^{(2)} & -\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{L}_b^{(2)} \\ \boldsymbol{0} & \boldsymbol{L}_b^{(1)^T} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{L}_b^{(2)^T} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_u^{(1)} \\ \boldsymbol{g}_b^{(1)} \\ \boldsymbol{q}_i^{(2)} \\ \boldsymbol{u}_b^{(2)} \\ \boldsymbol{g}_b^{(2)} \\ \boldsymbol{u}_\gamma \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_u^{(1)} \\ \bar{\boldsymbol{f}}_b^{(1)} \\ \boldsymbol{f}_i^{(2)} \\ \boldsymbol{f}_b^{(2)} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} \tag{13}$$

In this equation, the second row is the weak compatibility condition for component 1 while the fifth row can be recognized as the compatibility condition for component 2. The sixth row constitutes the equilibrium condition. The mass matrix is not shown for compactness; as usual it is simply the block diagonal of substructure mass matrices.

As in the previous sections, the three-field assembled equations of motion can be simplified to find a more compact expression. However, since we are considering mixed assembly, we cannot simply apply the primal or dual assembly methods of the previous sections. Instead, we should also mix the transformations. First it should be realized that in the case of mixed assembly we need both a unique interface displacement $\boldsymbol{u}_\gamma$ field and unique interface force field $\boldsymbol{\lambda}$ to facilitate the interaction between the force and displacement interface DOF of both substructures. One can then devise a transformation in the form:

$$\begin{bmatrix} \boldsymbol{q}_u^{(1)} \\ \boldsymbol{g}_b^{(1)} \\ \boldsymbol{q}_i^{(2)} \\ \boldsymbol{u}_b^{(2)} \\ \boldsymbol{g}_b^{(2)} \\ \boldsymbol{u}_\gamma \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{B}_b^{(1)^T} \\ \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{L}_b^{(2)} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{B}_b^{(2)^T} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_u^{(1)} \\ \boldsymbol{q}_i^{(2)} \\ \boldsymbol{u}_\gamma \\ \boldsymbol{\lambda} \end{bmatrix}$$

This transformation corresponds to primal assembly for the first structure as in the previous section, whereas the second component is subject to Dirichlet–Neumann assembly as outlined in Sec. II.A.3. Substitution of this transformation in Eq. (13) introduces both the unique interface force field and eliminates $\boldsymbol{u}_b^{(2)}$ (illustrated in Fig. 6). Premultiplication is needed only for the sake of symmetry and gives the mixed assembled system:

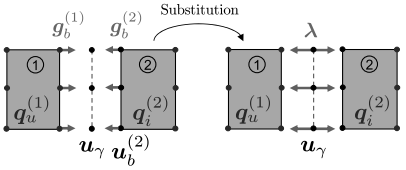

Fig. 6 Simplification of the three-field formulation of the mixed assembly case.

$$\boldsymbol{u}^{(s)} = \boldsymbol{R}^{(s)}\boldsymbol{q}^{(s)} \tag{17}$$

Here $\boldsymbol{R}^{(s)}$ represents the reduction basis of dimension $n^{(s)} \times r^{(s)}$; for efficient model reduction it should hold that $r^{(s)} \ll n^{(s)}$. Substituting the generalized DOF in Eq. (16) gives:

$$\boldsymbol{M}^{(s)}\boldsymbol{R}^{(s)}\ddot{\boldsymbol{q}}^{(s)} + \boldsymbol{K}^{(s)}\boldsymbol{R}^{(s)}\boldsymbol{q}^{(s)} = \boldsymbol{f}^{(s)} + \boldsymbol{g}^{(s)} + \boldsymbol{r}^{(s)} \tag{18}$$

$$\begin{bmatrix} \bar{\boldsymbol{M}}_{uu}^{(1)} & \boldsymbol{0} & \boldsymbol{0} & -\bar{\boldsymbol{M}}_{ug}^{(1)}\boldsymbol{B}_b^{(1)^T} \\ \boldsymbol{0} & \boldsymbol{M}_{ii}^{(2)} & \boldsymbol{M}_{ib}^{(2)}\boldsymbol{L}_b^{(2)} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{L}_b^{(2)^T}\boldsymbol{M}_{bi}^{(2)} & \boldsymbol{L}_b^{(2)^T}\boldsymbol{M}_{bb}^{(2)}\boldsymbol{L}_b^{(2)} & \boldsymbol{0} \\ -\boldsymbol{B}_b^{(1)}\bar{\boldsymbol{M}}_{gu}^{(1)} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{B}_b^{(1)}\bar{\boldsymbol{M}}_{gg}^{(1)}\boldsymbol{B}_b^{(1)^T} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{q}}_u^{(1)} \\ \ddot{\boldsymbol{q}}_i^{(2)} \\ \ddot{\boldsymbol{u}}_\gamma \\ \ddot{\boldsymbol{\lambda}} \end{bmatrix} + \cdots \begin{bmatrix} \bar{\boldsymbol{K}}_{uu}^{(1)} & \boldsymbol{0} & \boldsymbol{0} & -\bar{\boldsymbol{K}}_{ug}^{(1)}\boldsymbol{B}_b^{(1)^T} \\ \boldsymbol{0} & \boldsymbol{K}_{ii}^{(2)} & \boldsymbol{K}_{ib}^{(2)}\boldsymbol{L}_b^{(2)} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{L}_b^{(2)^T}\boldsymbol{K}_{bi}^{(2)} & \boldsymbol{L}_b^{(2)^T}\boldsymbol{K}_{bb}^{(2)}\boldsymbol{L}_b^{(2)} & \boldsymbol{L}_b^{(2)^T}\boldsymbol{B}_b^{(2)^T} \\ -\boldsymbol{B}_b^{(1)}\bar{\boldsymbol{K}}_{gu}^{(1)} & \boldsymbol{0} & -\boldsymbol{B}_b^{(1)}\boldsymbol{L}_b^{(1)} & \boldsymbol{B}_b^{(1)}\bar{\boldsymbol{K}}_{gg}^{(1)}\boldsymbol{B}_b^{(1)^T} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_u^{(1)} \\ \boldsymbol{q}_i^{(2)} \\ \boldsymbol{u}_\gamma \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_u^{(1)} \\ \boldsymbol{f}_i^{(1)} \\ \boldsymbol{L}_b^{(2)^T}\boldsymbol{f}_b^{(2)} \\ \boldsymbol{B}_b^{(1)}\bar{\boldsymbol{f}}_b^{(1)} \end{bmatrix} \tag{14}$$

Note that, as expected, the above mixed assembled system is symmetric, since we know that:

$$\sum_s \boldsymbol{B}_b^{(s)}\boldsymbol{L}_b^{(s)} = \sum_s (\,\boldsymbol{L}_b^{(s)^T} \quad \boldsymbol{B}_b^{(s)^T}\,)^T = \boldsymbol{0}$$

For the sake of illustration, let us now simplify the above assembled equations for the case where the boundary DOF of both substructures are ordered equally. In this case $\boldsymbol{L}_b^{(1)}$, $\boldsymbol{L}_b^{(2)}$, and $\boldsymbol{B}_b^{(2)}$ are identity matrices, while $\boldsymbol{B}_b^{(1)}$ is minus identity. If we reorder the DOF sets such that the Lagrange multipliers are associated to component 1 and the interface displacement field is associated to component 2, we can write the assembled equations of motion according to:

$$\begin{bmatrix} \bar{\boldsymbol{M}}_{uu}^{(1)} & \bar{\boldsymbol{M}}_{ug}^{(1)} & \boldsymbol{0} & \boldsymbol{0} \\ \bar{\boldsymbol{M}}_{gu}^{(1)} & \bar{\boldsymbol{M}}_{gg}^{(1)} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{M}_{ii}^{(2)} & \boldsymbol{M}_{ib}^{(2)} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{M}_{bi}^{(2)} & \boldsymbol{M}_{bb}^{(2)} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{q}}_u^{(1)} \\ \ddot{\boldsymbol{\lambda}} \\ \ddot{\boldsymbol{q}}_i^{(2)} \\ \ddot{\boldsymbol{u}}_\gamma \end{bmatrix} + \begin{bmatrix} \bar{\boldsymbol{K}}_{uu}^{(1)} & \bar{\boldsymbol{K}}_{ug}^{(1)} & \boldsymbol{0} & \boldsymbol{0} \\ \bar{\boldsymbol{K}}_{gu}^{(1)} & \bar{\boldsymbol{K}}_{gg}^{(1)} & \boldsymbol{0} & \boldsymbol{I} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{K}_{ii}^{(2)} & \boldsymbol{K}_{ib}^{(2)} \\ \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{K}_{bi}^{(2)} & \boldsymbol{K}_{bb}^{(2)} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_u^{(1)} \\ \boldsymbol{\lambda} \\ \boldsymbol{q}_i^{(2)} \\ \boldsymbol{u}_\gamma \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_u^{(1)} \\ \bar{\boldsymbol{f}}_b^{(1)} \\ \boldsymbol{f}_i^{(1)} \\ \boldsymbol{f}_b^{(2)} \end{bmatrix} \tag{15}$$

From the above equation one can clearly see the way the two systems interact. In addition to the external excitations, component 1 is excited by interface displacements from component 2 through its boundary DOF, while component 2 feels and additional forces from component 1 through its interface.

III. Review of Component Model Reduction Methods

In structural dynamics one frequently uses finite element models initially built for static analysis. These models are often very refined and often consist of several hundreds of thousands or even millions of DOF. In structural dynamic analyses (e.g., computing vibration modes, harmonic and/or transient responses) many staticlike problems need to be solved. An elegant approach would therefore be to reduce the number of DOF without modifying the mesh. Such methods are known as model reduction techniques, which employ the modal superposition principle to express physical displacements in terms of vibration mode shapes and modal amplitudes. In general, such methods start from the linear/linearized and discretized equations of motions of a substructure s:

$$\boldsymbol{M}^{(s)}\ddot{\boldsymbol{u}}^{(s)} + \boldsymbol{K}^{(s)}\boldsymbol{u}^{(s)} = \boldsymbol{f}^{(s)} + \boldsymbol{g}^{(s)} \tag{16}$$

As is usual in most CMS theory, damping is neglected here. The CMS methods then consist in transforming the original set of physical DOF $\boldsymbol{u}^{(s)}$ into a set of generalized DOF $\boldsymbol{q}^{(s)}$, as:

Since the transformation of Eq. (17) is an approximation, an error will usually be made on the dynamic equilibrium resulting in the residual force $\boldsymbol{r}^{(s)}$. By definition, this error is not in the space spanned by the reduction basis $\boldsymbol{R}$, and hence $\boldsymbol{R}^{(s)^T}\boldsymbol{r}^{(s)} = \boldsymbol{0}$. Using this property and projecting Eq. (18) onto the reduction basis one obtains the reduced equations of motion:

$$\tilde{\boldsymbol{M}}^{(s)}\ddot{\boldsymbol{q}}^{(s)} + \tilde{\boldsymbol{K}}^{(s)}\boldsymbol{q}^{(s)} = \tilde{\boldsymbol{f}}^{(s)} \tag{19}$$

Where the reduced substructure matrices are:

$$\begin{cases} \tilde{\boldsymbol{M}}^{(s)} = \boldsymbol{R}^{(s)^T}\boldsymbol{M}^{(s)}\boldsymbol{R}^{(s)} \\ \tilde{\boldsymbol{K}}^{(s)} = \boldsymbol{R}^{(s)^T}\boldsymbol{K}^{(s)}\boldsymbol{R}^{(s)} \\ \tilde{\boldsymbol{f}}^{(s)} = \boldsymbol{R}^{(s)^T}\boldsymbol{f}^{(s)} \end{cases} \tag{20}$$

Generally such a reduced representation is constructed for all the substructures in the system which are subsequently assembled, using the techniques discussed in the previous section, to obtain the reduced model of the complete structure. The "ingredients" in the reduction basis $\boldsymbol{R}$ are different for the various CMS methods, but in general they consist of vibration modes to account for the component dynamics and static modes to describe the interaction with neighboring substructures. In the remainder of this section, we will review the CB fixed interface method and more recent DCB free interface method. As will be shown, this DCB method is a more natural way of writing a free interface CMS method than the Rubin and MacNeal methods ([5,6]).

A. Classic Craig–Bampton Method

The Craig–Bampton method is based on the observation that the dynamic behavior of a subsystem can be described in terms of two types of information: 1) the *static constraint modes* resulting from unit displacements on the boundary DOF, and 2) the *internal vibration modes* found from fixing the boundary DOF. These ingredients form a statically complete reduction basis for the component [8]. To find this basis, start by partitioning the subsystem

DOF in a set of internal $\boldsymbol{u}_i$ and boundary $\boldsymbol{u}_b$ DOF:

$$\begin{bmatrix} \boldsymbol{M}_{bb} & \boldsymbol{M}_{bi} \\ \boldsymbol{M}_{ib} & \boldsymbol{M}_{ii} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{u}}_b \\ \ddot{\boldsymbol{u}}_i \end{bmatrix} + \begin{bmatrix} \boldsymbol{K}_{bb} & \boldsymbol{K}_{bi} \\ \boldsymbol{K}_{ib} & \boldsymbol{K}_{ii} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_b \\ \boldsymbol{u}_i \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_b \\ \boldsymbol{0} \end{bmatrix} + \begin{bmatrix} \boldsymbol{g}_b \\ \boldsymbol{0} \end{bmatrix} \quad (21)$$

Here the superscript s is omitted for compactness and it is assumed that no external excitations act on the internal DOF. The first step is to statically condense the internal DOF to the interface, corresponding to a Guyan reduction [19], by writing the second line in the equation above as

$$\boldsymbol{M}_{ib}\ddot{\boldsymbol{u}}_b + \boldsymbol{M}_{ii}\ddot{\boldsymbol{u}}_i + \boldsymbol{K}_{ib}\boldsymbol{u}_b + \boldsymbol{K}_{ii}\boldsymbol{u}_i = \boldsymbol{0} \quad (22)$$

subsequently neglecting the inertia forces and condensing the static part on the boundary DOF:

$$\boldsymbol{u}_i \approx -\boldsymbol{K}_{ii}^{-1}\boldsymbol{K}_{ib}\boldsymbol{u}_b = \boldsymbol{\Psi}_c\boldsymbol{u}_b \quad (23)$$

Here $\boldsymbol{\Psi}_c$ is the static condensation matrix containing the static constraint modes, which represent the static response of the internal DOF to unit displacements of the boundary DOF. To account for the component's dynamics, the Craig–Bampton method augments these with fixed interface vibration modes, found by setting $\boldsymbol{u}_b = \boldsymbol{0}$ in Eq. (21) and solving the eigenvalue problem:

$$(\boldsymbol{K}_{ii} - \omega_{i,j}\boldsymbol{M}_{ii})\phi_{i,j} = \boldsymbol{0} \quad (24)$$

Here ϕ_i is a single fixed interface vibration mode; the set of modes is denoted by $\boldsymbol{\Phi}_i$. Again, for efficient reduction it the number of fixed interface modes used in the reduction basis should be much smaller than the number of internal DOF. The Craig–Bampton reduction basis thus is:

$$\begin{bmatrix} \boldsymbol{u}_b \\ \boldsymbol{u}_i \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}_b \\ \boldsymbol{\Psi}_c\boldsymbol{u}_b + \boldsymbol{\Phi}_i\boldsymbol{\eta}_i \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{\Psi}_c & \boldsymbol{\Phi}_i \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_b \\ \boldsymbol{\eta}_i \end{bmatrix} = \boldsymbol{R}_{\mathrm{CB}} \begin{bmatrix} \boldsymbol{u}_b \\ \boldsymbol{\eta}_i \end{bmatrix} \quad (25)$$

Using this reduction matrix to reduce the original set of equations of motion in Eq. (21), we obtain the reduced mass and stiffness matrices $\tilde{\boldsymbol{M}}$ and $\tilde{\boldsymbol{K}}$. Using mass normalized vibration modes and mode orthogonality relationships, we can write for the reduced stiffness matrix:

$$\tilde{\boldsymbol{K}} = \begin{bmatrix} \tilde{\boldsymbol{K}}_{bb} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Omega}_i^2 \end{bmatrix} \quad \text{where } \tilde{\boldsymbol{K}}_{bb} = \boldsymbol{K}_{bb} - \boldsymbol{K}_{bi}\boldsymbol{K}_{ii}^{-1}\boldsymbol{K}_{ib} \quad (26)$$

while for the reduced mass matrix we find:

$$\tilde{\boldsymbol{M}} = \begin{bmatrix} \tilde{\boldsymbol{M}}_{bb} & \tilde{\boldsymbol{M}}_{b\phi} \\ \tilde{\boldsymbol{M}}_{\phi b} & \boldsymbol{I} \end{bmatrix}$$

$$\text{where } \tilde{\boldsymbol{M}}_{bb} = \boldsymbol{M}_{bb} - \boldsymbol{M}_{bi}\boldsymbol{K}_{ii}^{-1}\boldsymbol{K}_{ib} - \boldsymbol{K}_{bi}\boldsymbol{K}_{ii}^{-1}\boldsymbol{M}_{ib} + \boldsymbol{K}_{bi}\boldsymbol{K}_{ii}^{-1}\boldsymbol{M}_{ii}\boldsymbol{K}_{ii}^{-1}\boldsymbol{K}_{ib}$$

$$\tilde{\boldsymbol{M}}_{\phi b} = \boldsymbol{\Phi}_i^T(\boldsymbol{M}_{ib} - \boldsymbol{M}_{ii}\boldsymbol{K}_{ii}^{-1}\boldsymbol{K}_{ib}) = \tilde{\boldsymbol{M}}_{b\phi}^T$$

One of the strengths of the Craig–Bampton method is the straightforward calculation of its reduction basis. Secondly, the physical interface DOF $\boldsymbol{u}_b$ are retained in the reduced model, which facilitates easy assembly of the reduced substructures as *superelements* in common FE codes. A major drawback of the Craig–Bampton method is that if the substructure interface is changed the entire reduction basis needs to be recomputed.

B. Dual Craig–Bampton Method

After Craig and Bampton published their fixed interface CMS method in 1965, researchers were soon focussing on using free interface vibration modes in the reduction basis. This seems natural, as these are the "true" modes of the system and can be experimentally determined by testing the component in free–free conditions. These modes were first used in a CMS method by MacNeal in 1971 [5] and later by Rubin in 1975 [6]. Here, however, we focus on the DCB method introduced in 2004 [9] as an alternative to the original free interface CMS methods. As the name suggests, this method is the dual counterpart of the Craig–Bampton method, that is, the substructure models are reduced and assembled in a dual manner.[§] Again, two types of information are used for the reduction of the substructure models: 1) the *free interface vibration* modes of the structure to account for the dynamic behavior, and 2) the *residual flexibility modes* to account for the static response when excited at interface DOF.

To derive the DCB method, we start by writing the original set of DOF of a substructure as (again dropping the superscript s to denote a single substructure):

$$\boldsymbol{u} = \boldsymbol{u}_{\mathrm{stat}} + \sum_{j=r+1}^{n} \phi_j\boldsymbol{\eta}_j \quad (27)$$

where the total response of the substructure $\boldsymbol{u}$ is represented in terms of the free vibration modes of the substructure and a static solution. Here r is the number of rigid body modes of the substructure. The static response can be expressed as

$$\boldsymbol{u}_{\mathrm{stat}} = \boldsymbol{K}^{+}\boldsymbol{A}^T\boldsymbol{g}_b + \boldsymbol{\Phi}_r\boldsymbol{\eta}_r \quad (28)$$

where the first term describes the static flexible response to the interface forces and the second term gives the rigid body contribution ($\boldsymbol{\Phi}_r$ denotes the set of rigid body modes and $\boldsymbol{\eta}_r$ the associated amplitudes). Furthermore, $\boldsymbol{A}$ is a local Boolean matrix locating the interface DOF of the substructure and $\boldsymbol{g}_b$ are the interface (connection) forces from neighboring components. $\boldsymbol{K}^{+}$ is the generalized inverse of the stiffness matrix and is hence a flexibility matrix that we will subsequently denote by $\boldsymbol{G}$. If the substructure is constrained such that no rigid body modes exist ($r = 0$) then $\boldsymbol{G} = \boldsymbol{K}^{+} = \boldsymbol{K}^{-1}$. See [20] for more details. An approximation of the transformation of Eq. (27) is created by taking only the first $k \ll n$ free interface vibration modes:

$$\boldsymbol{u} \approx \boldsymbol{G}\boldsymbol{A}^T\boldsymbol{g}_b + \boldsymbol{\Phi}_r\boldsymbol{\eta}_r + \boldsymbol{\Phi}_f\boldsymbol{\eta}_f \quad (29)$$

These free interface vibration modes are simply found by solving the substructure's eigenvalue problem:

$$(\boldsymbol{K} - \omega_f^2\boldsymbol{M})\phi_f = \boldsymbol{0} \quad (30)$$

In this approximation the flexibility associated to the free vibration modes in $\boldsymbol{\Phi}_f$ is implicitly accounted for twice, since the spectral expansion of the flexibility matrix is:

$$\boldsymbol{G} = \sum_{j=r+1}^{n} \frac{\boldsymbol{\Phi}_j\boldsymbol{\Phi}_j^T}{\omega_j^2} \quad (31)$$

To simplify the expressions of the reduced system and obtain an $\boldsymbol{M}$ and $\boldsymbol{K}$ orthogonal basis, one could therefore subtract the flexibility that is already accounted for in the free vibration modes. As a result, the residual flexibility matrix is obtained (its properties are described in [9]):

$$\boldsymbol{G}_{\mathrm{res}} = \boldsymbol{G} - \sum_{j=r+1}^{k} \frac{\boldsymbol{\Phi}_j\boldsymbol{\Phi}_j^T}{\omega_j^2} \quad (32)$$

From the residual flexibility matrix the *residual flexibility modes* are found by simply picking the columns associated to the boundary DOF $\boldsymbol{u}_b$:

$$\boldsymbol{\Psi}_{\mathrm{ar}} = \boldsymbol{G}_{\mathrm{res}}\boldsymbol{A}^T \quad (33)$$

Substituting in Eq. (29) leads to the final transformation for $\boldsymbol{u}$ (with $\boldsymbol{\Phi} = [\,\boldsymbol{\Phi}_r \quad \boldsymbol{\Phi}_f\,]$):

$$\boldsymbol{u} = \boldsymbol{\Psi}_{\mathrm{ar}}\boldsymbol{g}_b + \boldsymbol{\Phi}_r\boldsymbol{\eta}_r + \boldsymbol{\Phi}_f\boldsymbol{\eta}_f = \boldsymbol{\Psi}_{\mathrm{ar}}\boldsymbol{g}_b + \boldsymbol{\Phi}\boldsymbol{\eta} \quad (34)$$

[‡]Note that although the name "dual Craig–Bampton" was coined in [9], a similar but less general method was also discussed in [13].

The full set of DOF is thus written in terms of residual attachment modes $\boldsymbol{\Psi}_{\text{ar}}$, rigid body modes $\boldsymbol{\Phi}_r$, and free interface vibration modes $\boldsymbol{\Phi}_f$. Rewriting the reduction basis into a matrix-vector form gives:

$$\begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{g}_b \end{bmatrix} = \boldsymbol{R}_{\text{DCB}} \begin{bmatrix} \boldsymbol{\eta} \\ \boldsymbol{g}_b \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Phi} & \boldsymbol{\Psi}_{\text{ar}} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta} \\ \boldsymbol{g}_b \end{bmatrix} \tag{35}$$

In addition to the original substructure displacements $\boldsymbol{u}$ the interface forces $\boldsymbol{g}_b$ are included in the new set of DOF. The substructure's equation of motion can thus be written as:

$$\begin{bmatrix} \boldsymbol{M} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{u}} \\ \ddot{\boldsymbol{g}}_b \end{bmatrix} + \begin{bmatrix} \boldsymbol{K} & -\boldsymbol{A}^T \\ -\boldsymbol{A} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{g}_b \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{0} \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} \\ -\boldsymbol{u}_b \end{bmatrix} \tag{36}$$

The second equation seems redundant, but is added for symmetry and is used to enforce compatibility during assembly. By projecting these equations onto the reduction basis and using the properties of the residual flexibility matrix, the following reduced matrices are found:

$$\begin{aligned} \bar{\boldsymbol{M}} &= \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \bar{\boldsymbol{M}}_{\text{res}} \end{bmatrix} \quad \text{with } \bar{\boldsymbol{M}}_{\text{res}} = \boldsymbol{\Psi}_{\text{ar}}^T \boldsymbol{M} \boldsymbol{\Psi}_{\text{ar}} \\ \bar{\boldsymbol{K}} &= \begin{bmatrix} \boldsymbol{\Omega}^2 & -\boldsymbol{\Phi}^T \boldsymbol{A}^T \\ -\boldsymbol{A}\boldsymbol{\Phi} & -\bar{\boldsymbol{F}}_{\text{res}} \end{bmatrix} \quad \text{with } \bar{\boldsymbol{F}}_{\text{res}} = \boldsymbol{A} \boldsymbol{\Psi}_{\text{ar}} \end{aligned} \tag{37}$$

Here $\boldsymbol{\Omega}^2$ is the diagonal matrix of eigenfrequencies and $\bar{\boldsymbol{F}}_{\text{res}}$ is a true flexibility matrix, since:

$$\bar{\boldsymbol{F}}_{\text{res}} = \boldsymbol{A} \boldsymbol{\Psi}_{\text{ar}} = \boldsymbol{A} \boldsymbol{G}_{\text{res}} \boldsymbol{A}^T$$

The major advantage of the DCB method is that if the interface is altered only the set of residual flexibility modes has to be recomputed (i.e., replacing, removing or adding columns of the residual flexibility matrix). Furthermore, experimental results can be used to validate the reduced model, since experimentally determined mode shapes are usually free interface modes. On the downside, assembly of the substructures is less straightforward and involves connecting interface forces, as discussed in Sec. II.B.

The same ingredients for component reduction were already proposed by MacNeal [5] and Rubin [6], but there is an important difference. The Rubin and MacNeal methods transform the interface force DOF back to interface displacement DOF to artificially enable assembly as in Secs. II and II.A hence the reduced component can be treated as a superelement. This, however, leads to reduced matrices that are not block diagonal and less sparse. The current method, however, keeps the interface forces as part of the new set of generalized DOF. The substructures are assembled using the interface forces and thereby satisfy the interface equilibrium exactly but compatibility only in a weak sense, whereas in the Craig–Bampton, MacNeal, or Rubin methods the compatibility is exact while the equilibrium is weakened. The weakening of the compatibility can be seen from the second line in Eq. (37):

$$\boldsymbol{A}\left(\boldsymbol{\Phi}\boldsymbol{\eta} + \boldsymbol{\Psi}_{\text{ar}} \boldsymbol{g}_b\right) = \boldsymbol{u}_b - \bar{\boldsymbol{M}}_{\text{res}} \ddot{\boldsymbol{g}}_b$$

If one neglects the last term (i.e., $\bar{\boldsymbol{M}}_{\text{res}} = \boldsymbol{0}$), exact compatibility is found (as in the MacNeal method [5]) since $\boldsymbol{A}\boldsymbol{u} = \boldsymbol{u}_b$. The $\bar{\boldsymbol{M}}_{\text{res}}$ term actually allows a small error on the compatibility equation, namely $\boldsymbol{A}\boldsymbol{u} = \boldsymbol{u}_b + \boldsymbol{\epsilon}$, due to the modal mass and flexibility on the interface associated with the eigenmodes that were discarded during reduction. This allows an incompatibility between the substructures in the space spanned by the discarded modes. If the reduction basis is too poor, the incompatibility can introduce spurious modes and eigenfrequencies in the frequency range of interest. By enriching the reduction basis with more eigenmodes and/or with higher-order residual modes as described in [21], these spurious modes can be shifted to higher frequencies.

IV. Generalization: Mixed Craig–Bampton Method

In the previous section, the CB and DCB methods were reviewed. This discussion paved the way to introduce in this section the novel MCB method which, as we will show later, is a natural mixed boundary generalization of the two previous methods. In general, the MCB method comprises the following steps:

1) For all components in the assembly define the interfaces.

2) Using some criterion, choose boundary conditions for all pairs of interface DOF.

3) Calculate the reduction bases for the components and compute the reduced matrices.

4) Assemble the reduced components in the correct manner.

The first step is no different than for other CMS methods. The subsequent steps, however, are nonstandard and will be discussed next. It should be noted that the reduction of substructures using the MCB method cannot be performed separately, since the reduction basis of one substructure is dependent on the properties of its neighboring components.

A. Selection of Free or Fixed Modes

To select fixed or free modes for the reduction basis of the components, some criterion must be established. To this end, an a priori estimate is needed of the components' behavior; ideally the responses of all component to a unit load or displacement at all their interface DOF. This is, however, computationally inefficient, so an approximation can be made by estimating the substructure behavior by looking only at the value on the diagonal of the stiffness matrix corresponding to the interface DOF. Then, three cases can be distinguished and the following selection scheme is proposed:

1) In the first case, subsystem 1 is much stiffer than subsystem 2: $K_{ii}^{(1)}/K_{ii}^{(2)} > 10^c$, where c is some constant that can be chosen to suit the problem at hand. In this case, subsystem 1 will feel some connection forces through its interface but will not be influenced much by the presence of its neighboring substructure, behaving nearly as if it were free. The motion of component 2 will, however, be largely dictated through its interface with component 1. The natural choice would thus be to let free the interface DOF of component 1 in the reduction, while the corresponding interface DOF of substructure 2 should be fixed. We will denote DOF that remain free by "dual" DOF, while fixed DOF we call "primal."

2) In the second case, the stiffness at the interface DOF of both subsystems is of approximately the same order of magnitude, i.e., $10^{-c} \leq K_{ii}^{(1)}/K_{ii}^{(2)} \leq 10^c$. In this case, both interface DOF can be reduced with either fixed or free interface modes. The choice for fixed or free modes can be made per set of interface DOF, although a consistent choice for the complete assembly leads to a simpler assembly procedure.

3) In the third case, subsystem 2 is much stiffer then 1: $K_{ii}^{(1)}/K_{ii}^{(2)} < 10^{-c}$. Using the same reasoning as before, the natural choice is to reduce subsystem 2 with free interface modes and system 1 with fixed interface modes.

So, the goal of the selection scheme is to use some (simple) knowledge of the assembled system to construct a better reduction basis for the components. As such, the method is somewhat similar to the application of Robin-type interface conditions (see, e.g., [22]), where stiffness and mass of the neighboring substructures is in some form taken into account in the component reduction. Thereby the modes used in the component reduction bases more closely represent the assembled mode shapes.

Using the above selection scheme, the substructure DOF vector $\boldsymbol{u}^{(s)}$ can be partitioned into internal DOF $\boldsymbol{u}_i^{(s)}$, dual DOF $\boldsymbol{u}_d^{(s)}$, and "primal" DOF $\boldsymbol{u}_p^{(s)}$, as:

$$\boldsymbol{u}^{(s)} = \begin{bmatrix} \boldsymbol{u}_i^{(s)} & \boldsymbol{u}_d^{(s)} & \boldsymbol{u}_p^{(s)} \end{bmatrix}^T \tag{38}$$

We can now introduce the DOF set $\boldsymbol{u}_m^{(s)}$, the set of internal plus dual DOF, to denote the DOF that will be replaced by generalized DOF in the reduction:

$$\boldsymbol{u}_m^{(s)} = [\,\boldsymbol{u}_i^{(s)} \quad \boldsymbol{u}_d^{(s)}\,]^T \tag{39}$$

The above division of DOF will be used in the next subsection to find the reduction basis. It should be remarked that the above proposed selection method can be easily automated but is useful only if the component interface coincides with the material interface. If this is not the case, the values on the diagonal of the stiffness matrix not truly reflect the "global" stiffness of the system (imagine for instance a rubber bushing with a metal core). In such situations one should use some other criterion for selecting fixed/free modes or resort to "engineering judgement."

B. Reduction of Subsystems

Given the partitioning of substructure DOF introduced above, the partitioned equations of motion of a substructure become (again the substructure denotation superscript (s) is omitted for clarity):

$$\begin{bmatrix} \boldsymbol{M}_{mm} & \boldsymbol{M}_{mp} \\ \boldsymbol{M}_{pm} & \boldsymbol{M}_{pp} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{u}}_m \\ \ddot{\boldsymbol{u}}_p \end{bmatrix} + \begin{bmatrix} \boldsymbol{K}_{mm} & \boldsymbol{K}_{mp} \\ \boldsymbol{K}_{pm} & \boldsymbol{K}_{pp} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_m \\ \boldsymbol{u}_p \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_m \\ \boldsymbol{f}_p \end{bmatrix} + \begin{bmatrix} \boldsymbol{g}_m \\ \boldsymbol{g}_p \end{bmatrix} \tag{40}$$

It should now be realized that the DOF in $\boldsymbol{u}_m$ will be reduced and assembly of the $\boldsymbol{u}_d$ in this DOF set will be performed using interface forces. Hence, the interface forces $\boldsymbol{g}_d$ need to be included in the DOF vector and an additional equation is added to ensure symmetry of the equations:

$$\begin{bmatrix} \boldsymbol{M}_{mm} & \boldsymbol{M}_{mp} & \boldsymbol{0} \\ \boldsymbol{M}_{pm} & \boldsymbol{M}_{pp} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{u}}_m \\ \ddot{\boldsymbol{u}}_p \\ \ddot{\boldsymbol{g}}_d \end{bmatrix} + \begin{bmatrix} \boldsymbol{K}_{mm} & \boldsymbol{K}_{mp} & -\boldsymbol{A}_m^T \\ \boldsymbol{K}_{pm} & \boldsymbol{K}_{pp} & \boldsymbol{0} \\ -\boldsymbol{A}_m & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_m \\ \boldsymbol{u}_p \\ \boldsymbol{g}_d \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_m \\ \boldsymbol{f}_p \\ \boldsymbol{0} \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{g}_p \\ -\boldsymbol{u}_d \end{bmatrix} \tag{41}$$

Here $\boldsymbol{A}_m$ is the Boolean matrix localizing the DOF $\boldsymbol{u}_d$ in $\boldsymbol{u}_m$. As a consequence of the partitioning of DOF, the MCB reduction base in general consists of three ingredients: 1) static constraint modes $\boldsymbol{\Psi}_c$ associated to the interface DOF $\boldsymbol{u}_p$ that are fixed, 2) residual flexibility modes $\boldsymbol{\Psi}_{\text{ar}}$ associated to the interface DOF $\boldsymbol{u}_d$ that will be left free, and 3) a truncated set of fixed/free vibration modes $\boldsymbol{\Phi}_m$ of the structure

$$\boldsymbol{u}_m \approx \boldsymbol{\Phi}_m \boldsymbol{\eta}_m + \boldsymbol{\Psi}_{\text{ar}} \boldsymbol{g}_d + \boldsymbol{\Psi}_c \boldsymbol{u}_p \tag{42}$$

The ingredients of the reduction basis can be computed as follows. Firstly, the vibration modes result from solving the fixed/free eigenproblem with the DOF in $\boldsymbol{u}_m$ free and the $\boldsymbol{u}_p$ fixed, so:

$$(\boldsymbol{K}_{mm} - \omega^2 \boldsymbol{M}_{mm})\boldsymbol{\Phi}_m = \boldsymbol{0}$$

Note that in case the fixed DOF in $\boldsymbol{u}_p$ do not fully constrain the system, $\boldsymbol{\Phi}_m$ also contains the remaining rigid body modes. Secondly, the constraint modes can be computed by condensing the stiffness matrix to the primal DOF, as:

$$\boldsymbol{\Psi}_c = -\boldsymbol{K}_{mm}^{+} \boldsymbol{K}_{mp}$$

Note that in case a set of primal interface DOF is chosen that constrains the rigid body modes of the substructure, the pseudoinverse superscript $+$ becomes a normal inverse. Finally, the residual flexibility modes can be found by:

$$\boldsymbol{\Psi}_{\text{ar}} = \boldsymbol{K}_{mm}^{+} \boldsymbol{A}_m^T - \sum_{i=n_r+1}^{n} \frac{\boldsymbol{\Phi}_{m,i} \boldsymbol{\Phi}_{m,i}^T \boldsymbol{A}_m^T}{\omega_{m,i}^2} \tag{43}$$

Only the contribution of the flexible modes is taken into account; the possible n_r rigid body modes in $\boldsymbol{\Phi}_m$ do not contribute to the stiffness. Again it should be noted that the pseudoinverse becomes a normal inverse in case there are no rigid body modes in $\boldsymbol{K}_{mm}$. Next, the MCB reduction matrix can be put in matrix form as:

$$\begin{bmatrix} \boldsymbol{u}_m \\ \boldsymbol{u}_p \\ \boldsymbol{g}_p \end{bmatrix} = \boldsymbol{R}_{\text{MCB}} \begin{bmatrix} \boldsymbol{\eta}_m \\ \boldsymbol{u}_p \\ \boldsymbol{g}_p \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Phi}_m & \boldsymbol{\Psi}_c & \boldsymbol{\Psi}_{\text{ar}} \\ \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta}_m \\ \boldsymbol{u}_p \\ \boldsymbol{g}_p \end{bmatrix} \tag{44}$$

Using this reduction basis, the reduced stiffness matrix can be computed as:

$$\tilde{\boldsymbol{K}} = \begin{bmatrix} \boldsymbol{\Omega}_m^2 & \boldsymbol{0} & -\boldsymbol{\Phi}_{m,m}^T \\ \boldsymbol{0} & \tilde{\boldsymbol{K}}_{pp} & -\boldsymbol{\Psi}_{C,m}^T \\ -\boldsymbol{\Phi}_{m,m} & -\boldsymbol{\Psi}_{C,m} & -\bar{\boldsymbol{F}}_{\text{res}} \end{bmatrix}$$

$$\text{where } \tilde{\boldsymbol{K}}_{pp} = \boldsymbol{K}_{pp} - \boldsymbol{K}_{pm} \boldsymbol{K}_{mm}^{+} \boldsymbol{K}_{mp}$$

$$\bar{\boldsymbol{F}}_{\text{res}} = \boldsymbol{A}_m \boldsymbol{\Psi}_{\text{ar}} \qquad \boldsymbol{\Phi}_{m,m} = \boldsymbol{A}_m \boldsymbol{\Phi}_m \qquad \boldsymbol{\Psi}_{C,m} = \boldsymbol{A}_m \boldsymbol{\Psi}_c$$

and $\boldsymbol{\Omega}_m^2$ is a diagonal matrix containing the squares of the fixed/free eigenfrequencies of the system. For the reduced mass matrix we find:

$$\tilde{\boldsymbol{M}} = \begin{bmatrix} \boldsymbol{I} & \tilde{\boldsymbol{M}}_{\phi m} & \boldsymbol{0} \\ \tilde{\boldsymbol{M}}_{m\phi} & \tilde{\boldsymbol{M}}_{pp} & \boldsymbol{M}_{m\psi} \\ \boldsymbol{0} & \boldsymbol{M}_{\psi m} & \bar{\boldsymbol{M}}_{\text{res}} \end{bmatrix} \quad \text{with } \bar{\boldsymbol{M}}_{\text{res}} = \boldsymbol{\Psi}_{\text{ar}}^T \boldsymbol{M}_{mm} \boldsymbol{\Psi}_{\text{ar}}$$

$$\tilde{\boldsymbol{M}}_{\phi m} = \boldsymbol{M}_{m\phi}^T = \boldsymbol{\Phi}_m^T (\boldsymbol{M}_{mp} - \boldsymbol{M}_{mm} \boldsymbol{K}_{mm}^{+} \boldsymbol{K}_{mp})$$

$$\boldsymbol{M}_{\psi m} = \boldsymbol{M}_{m\psi}^T = \boldsymbol{\Psi}_{\text{ar}}^T (\boldsymbol{M}_{mp} - \boldsymbol{M}_{mm} \boldsymbol{K}_{mm}^{+} \boldsymbol{K}_{mp})$$

$$\tilde{\boldsymbol{M}}_{pp} = \boldsymbol{M}_{pp} + \boldsymbol{\Psi}_c^T \boldsymbol{M}_{mm} \boldsymbol{\Psi}_c - \boldsymbol{M}_{pm} \boldsymbol{\Psi}_c - \boldsymbol{\Psi}_c^T \boldsymbol{M}_{mp}$$

Assembly of systems reduced using the MCB method is done using the mixed assembly procedure as outlined in Secs. II and II.C. From the above equation one can clearly see that the MCB method is a true generalization of the original Craig–Bampton and DCB methods; if there are no dual DOFs (i.e., $\boldsymbol{u}_d$ is empty) the reduced matrices are exactly equal to those found with the Craig–Bampton method whereas in the absence of primal DOF (i.e., $\boldsymbol{u}_p$ is empty) the matrices of the DCB method are found. Practically, one can implement this by taking a high value for the parameter c in the selection scheme. The scheme will then consider the stiffness of all connected pairs of interface DOF to be of the same magnitude and hence an equal boundary condition will be chosen on both sides.

Note that this reduction basis in Eq. (44) was already proposed in [13], where a mixed assembly was also proposed but was not general and limited to decompositions where no more then two substructures connect on an interface. Also, this basis was later proposed in [11], but there, very much like in the MacNeal and Rubin method, the authors eliminated the interface forces from the basis, leading to full and intricate reduced matrices.

V. Case Study

In this section the mixed reduction method is applied to the test problem shown in Fig. 7 consisting of three substructures: the horizontal (A), vertical (B), and diagonal (C) solid beams, all having two interfaces. The horizontal and vertical beam have the same properties; the diagonal beam is lighter and more flexible:

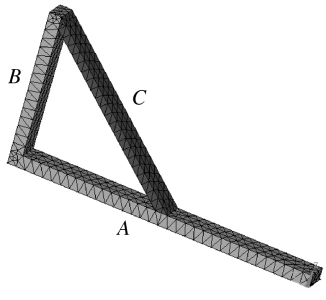

Fig. 7 CMS test problem.

1) Beams A and B: density 7850 (kg/m^3), Young's modulus 210 (GPa), Poisson ratio 0.3.

2) Beam C: density 2200 (kg/m^3), Young's modulus 4.2 (GPa), Poisson ratio 0.3.

The parameter c in the selection scheme has been chosen equal to one, meaning that when a stiffness ratio of more than ten is found the scheme selects different boundary conditions for both DOF. For interface DOF pairs of equal stiffness fixed interface (i.e., primal) boundary conditions were chosen. The mesh has been created using ten-node tetrahedral solid elements resulting in a full model of almost 11,000 DOF. To assess the performance of the MCB method with respect to the existing methods, we created reduced models using the classic, dual and mixed Craig–Bampton methods and compared these to the full model. First, reduced models were created using 15 vibration modes per substructure (including the rigid body modes for the dual and possibly MCB methods); to test the convergence of the methods this analysis was repeated employing 30 vibration modes per substructure.

For the comparison, a modal analysis has been performed on all models and the frequency error and mode shape error with respect to the full solution were determined. The latter was expressed using the modal assurance criterion (MAC) [23] and computing the 1-MAC values (i.e., the error on the mode shape correlation) for each corresponding mode pair. The results are presented in Fig. 8 for the case using 15 modes and Fig. 9 for the case using 30 modes. Note that since all methods can predict the rigid body motions exactly, only the errors on the flexible modes are plotted and hence the mode number on the y-axis starts at seven.

From the results in Fig. 8, we see that the DCB model performs best in the low-frequency range. It is believed that this is due to the fact that, in addition to the static (residual attachment) modes, rigid body modes are explicitly present in the reduction bases, while in the CB model the rigid body motions are described by the static constraint modes. Consequently the DCB model has six more generalized DOF to describe the low-frequency behavior, leading to smaller errors with respect to the full model. Since the MCB model has no rigid body modes in the bases (due to the choice of fixed boundary conditions for equal stiffness interfaces) it behaves similar to the CB model in the low-frequency range.

In the higher-frequency range, however, the CB and MCB models perform better. Because of the absence of rigid body modes in their reduction bases, more vibrational information is included. This allows for a better representation of the higher-order global mode shapes. The accuracy of the MCB model is a little better than the CB model, and we believe this is due to the fact that the MCB reduction base is constructed using already some knowledge of the assembled system.

When the reduction bases of all models are enriched to include 30 vibration modes, the results shown in Fig. 9 are obtained. First, we see that all three reduced models show much smaller errors on the eigenfrequencies and modes compared with before, indicating that the methods indeed converge. Secondly, the comments made above still largely hold although the DCB model is much more competitive in this case. Furthermore, we see that the MCB model is slightly more accurate than the CB model at low frequencies, while it is better then the DCB model at higher modes. The overall accuracy of the MCB model is thus in between the CB and DCB models, emphasizing that the MCB method is truly a generalization of the other methods.

To conclude this section, it should be noted that the current case study is by no means exhaustive. It gives a good indication of the performance of the MCB method, but only by applying the method to more and different problems more experience and insight can be gained into its true merits. Nonetheless the results presented here look promising.

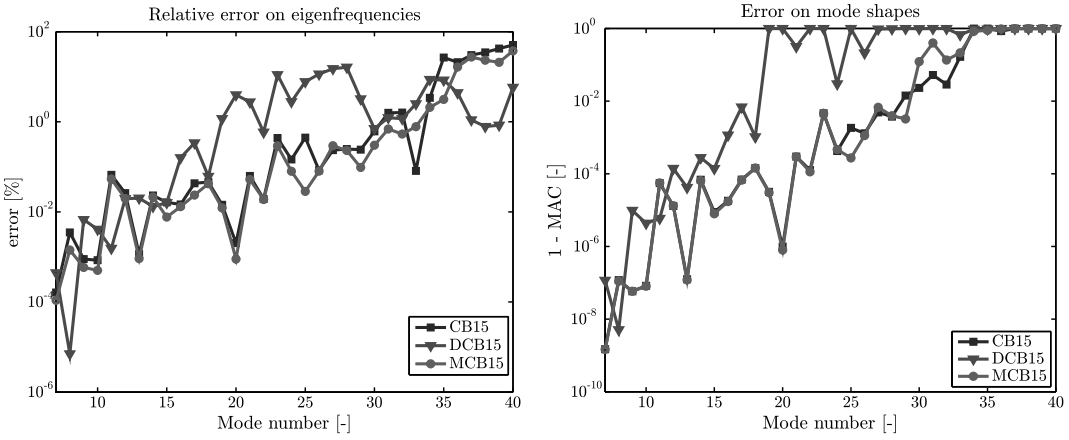

Fig. 8 Comparison of CMS methods using 15 vibration modes.

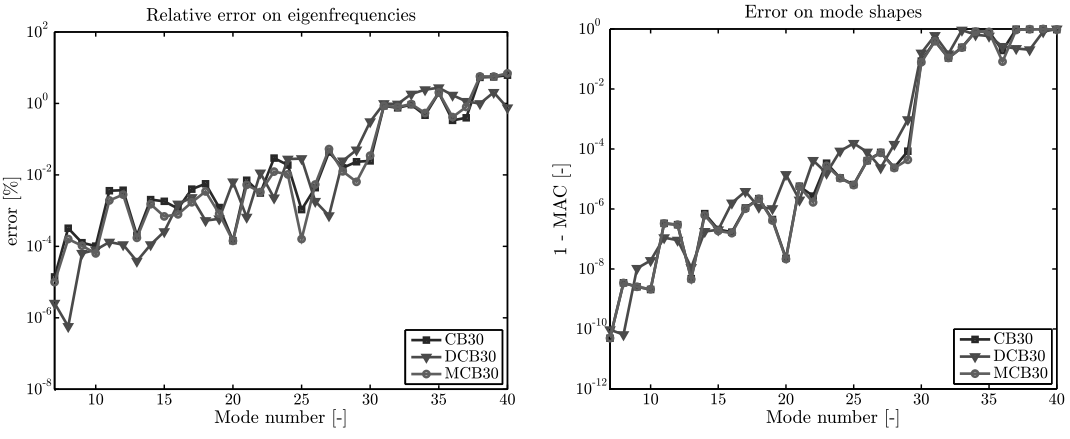

Fig. 9 Comparison of CMS methods using 30 vibration modes.

VI. Conclusions

In this paper a general approach was proposed to component model assembly and reduction for dynamic substructuring by considering these aspects separately. First, a general framework for assembly of (reduced) component models was proposed, which enables assembly of all types of component models. Thereby free-mode reduced models can be directly assembled to any other (reduced) component model, eliminating the need to transform the interface force DOF back to displacement DOF as is classically done. Secondly, a mixed boundary model reduction method was presented. The combined methods allow to tackle mixed boundary problems in a consistent manner, and whereas other mixed boundary reduction methods often result in fully populated reduced matrices, the current approach gives sparse matrices. It was shown that, theoretically, the proposed methodology is a generalization of the CCB and more recent DCB methods.

Application to a test problem revealed that the current method is at least as accurate as the existing methods but has superior versatility. Hence it can be concluded that the proposed methodology can be a valuable tool for solving substructured dynamic problems, for instance involving components with large relative differences in stiffness and/or mass.

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